

Helly EPT graphs on bounded degree trees: forbidden induced subgraphs and efficient recognition

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Abstract

The edge intersection graph of a family of paths in host tree is called an *EPT* graph. When the host tree has maximum degree h , we say that G belongs to the class $[h, 2, 2]$. If, in addition, the family of paths satisfies the Helly property, then $G \in \text{Helly } [h, 2, 2]$. The time complexity of the recognition of the classes $[h, 2, 2]$ inside the class *EPT* is open for every $h > 4$. In [6], Golumbic et al. wonder if the only obstructions for an *EPT* graph belonging to $[h, 2, 2]$ are the chordless cycles C_n for $n > h$. In the present paper, we give a negative answer to that question, we present a family of *EPT* graphs which are forbidden induced subgraphs for the classes $[h, 2, 2]$. Using them we obtain a total characterization by induced forbidden subgraphs of the classes $\text{Helly } [h, 2, 2]$ for $h \geq 4$ inside the class *EPT*. As a byproduct, we prove that $\text{Helly } EPT \cap [h, 2, 2] = \text{Helly } [h, 2, 2]$. Following the approach used in [10], we characterize $\text{Helly } [h, 2, 2]$ graphs by their atoms in the decomposition by clique separators. We give an efficient algorithm to recognize $\text{Helly } [h, 2, 2]$ graphs.

Keywords: intersection graphs, EPT graphs, UE graphs, tolerance graphs.

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1 Introduction

A graph G is called *EPT* (or *UE*) if it is the edge intersection graph of a family of paths in a tree. *EPT* graphs are used in network applications, the problem of scheduling undirected calls in a tree network is equivalent to the problem of coloring an *EPT* graph (see [2]). The class of *EPT* graphs was first investigated by Golumbic and Jamison [3,4]. In the last decades many papers were devoted to the study of *EPT* graphs and their generalizations, see [5,8,11]. In [9], the class of graphs that admit an *EPT* representation on a host tree with maximum degree h is denoted by $[h, 2, 2]$. Clearly, $[2, 2, 2]$ is the class of interval graphs. It is known that $[3, 2, 2]$ is precisely the class of chordal *EPT* graphs [9], while $[4, 2, 2]$ is the class of weakly chordal *EPT* graphs [7]. Notice that the class of *EPT* graphs is the union of the classes $[h, 2, 2]$ for $h \geq 2$. A complete hierarchy of related graph classes emerging by imposing different restrictions on the tree representation is published in [6].

On the algorithmic side, the recognition and coloring problems restricted to *EPT* graphs are NP-complete, whereas the maximum clique and maximum stable set problems are polynomially solvable. See [3].

The time complexity of the recognition of the classes $[h, 2, 2]$ inside the class *EPT* is open for $h > 4$, and it is known to be polynomial time solvable for $h \in \{2, 3, 4\}$. In [6] and [7], Golumbic et al. wonder if the only obstructions for an *EPT* graph belonging to $[h, 2, 2]$ are the chordless cycles of size greater than h . In [1], we give a negative answer to this question and present a family of forbidden induced subgraphs called prisms.

In this paper, we generalize the class of prisms and present a wider family of *EPT* graphs called k -gates which are forbidden induced subgraphs for the classes $[h, 2, 2]$ when $h < k$.

A graph is Helly *EPT* (or *UEH*) if it admits an *EPT* representation using a path family that satisfies the Helly property. In [10], Monma and Wei characterize *EPT* and Helly *EPT* via decomposing the graph by clique separators and prove that the latter class can be recognized efficiently. Finding a characterization by forbidden induced subgraphs of *EPT* and of Helly *EPT* graphs are long standing open problems.

Helly $[h, 2, 2]$ is the class of graphs that admit a Helly *EPT* representation on a host tree with maximum degree h . Clearly, $\text{Helly } EPT \cap [h, 2, 2] \subseteq \text{Helly } [h, 2, 2]$ but the equality not necessary holds.

We obtain a total characterization by induced forbidden subgraphs of the class Helly $[h, 2, 2]$ inside the class *EPT* using gates. As a byproduct, we prove that $\text{Helly } EPT \cap [h, 2, 2] = \text{Helly } [h, 2, 2]$ which means that, in the

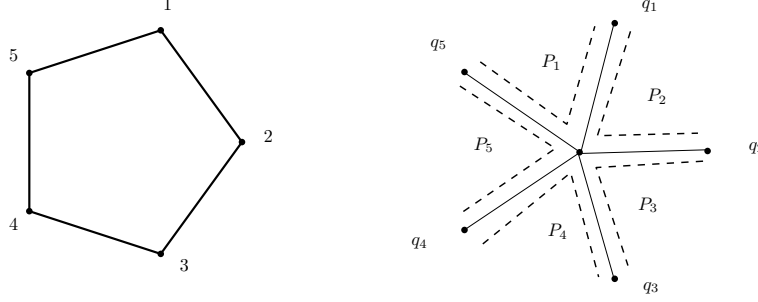


Fig. 1. The cycle C_5 and its EPT representation: a pie of size 5.

way of way of getting a Helly representation, it is not necessary to increase the maximum degree of the host tree.

In addition, we characterize Helly $[h, 2, 2]$ graphs by their atoms in the decomposition by clique separators. We give an efficient algorithm to recognize Helly $[h, 2, 2]$ graphs.

The paper is organized as follows: in Section 2, we provide basic definitions and known results. In Section 3, we depict the graphs named k -gates and focus on their main properties; we show that k -gates are Helly EPT but do not admit an EPT representation on a host tree with maximum degree less than k . In Section 4, we show that a Helly EPT graph G belongs to the class Helly $[h, 2, 2]$ if and only if G does not have a k -gate as induced subgraph for any $k > h$. Finally, in Section 5, we use the Monma and Wei decomposition by clique separator to obtain an efficient algorithm for the recognition of Helly $[h, 2, 2]$ graphs.

2 Preliminaries and known results

In this paper all graphs are finite and simple. Given a graph G , $V(G)$ and $E(G)$ denote the vertex set and the edge set of G , respectively. An **EPT representation** of G is a pair $\langle \mathcal{P}, T \rangle$ where \mathcal{P} is a family $(P_v)_{v \in V(G)}$ of subpaths of the **host tree** T satisfying that two vertices v and w of G are adjacent if and only if $E(P_v) \cap E(P_w) \neq \emptyset$. When the maximum degree of the host tree T is h , the EPT representation of G is called an **(h, 2, 2)-representation** of G . The class of graphs that admit an $(h, 2, 2)$ -representation is denoted by $[\mathbf{h}, 2, 2]$.

A **star** is any complete bipartite graph $K_{1,n}$. The only vertex with degree greater than one is called the **center of the star**. The edges of a star are called **spokes**. The star $K_{1,3}$ is named the **claw graph**. We will say that a path $P : (v_1, \dots, v_l)$ contains a vertex v if $v = v_i$ for some $1 \leq i \leq l$; and that it contains an edge e if $e = v_i v_{i+1}$ for some $1 \leq i \leq l - 1$.

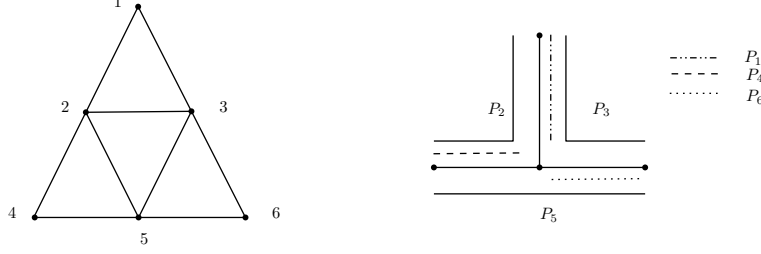


Fig. 2. An *EPT* representation of the sun S_3 . In this representation, the central triangle $\{2, 3, 5\}$ is a claw-clique; the other three triangles are edge-cliques.

Golumbic et al. introduced the notion of pie in order to describe *EPT* representations of chordless cycles. A **pie of size k** in an *EPT* representation $\langle \mathcal{P}, T \rangle$ is a star subgraph of T with central vertex q and neighbors q_1, \dots, q_k and a subfamily of paths P_1, \dots, P_k of \mathcal{P} such that $\{q_i, q, q_{i+1}\} \subseteq V(P_i)$, for $1 \leq i \leq k$ (addition is assumed to be module n). See Figure 1.

Theorem 1 [3] *Let $\langle \mathcal{P}, T \rangle$ be an *EPT* representation of a graph G . If G contains a chordless cycle C_k with $k \geq 4$, then $\langle \mathcal{P}, T \rangle$ contains a pie of size k whose paths are in one-to-one correspondence with the vertices of C_k .*

A set family $(S_i)_{i \in I}$ satisfies the **Helly property** if any pairwise intersecting subfamily $(S_i)_{i \in I'}$ with $\emptyset \neq I' \subseteq I$ has non-empty total intersection, i.e. $\bigcap_{i \in I'} S_i \neq \emptyset$. A graph G is **Helly EPT** if it admits an *EPT* representation $\langle \mathcal{P}, T \rangle$ such that the set family $(E(P))_{P \in \mathcal{P}}$ satisfies the Helly property. In an analogous way, we say that G is **Helly $[h, 2, 2]$** if it admits an $(h, 2, 2)$ -representation $\langle \mathcal{P}, T \rangle$ such that the family $(E(P))_{P \in \mathcal{P}}$ satisfies the Helly property. Clearly, $\text{Helly } [h, 2, 2] \subseteq \text{Helly EPT} \cap [h, 2, 2]$.

A **complete set** of a graph G is a subset of $V(G)$ whose elements are pairwise adjacent. A **clique** is a maximal (with respect to the inclusion relation) complete set.

Given an *EPT* representation $\langle (P_v)_{v \in V(G)}, T \rangle$ of G , for every edge e of T , let K_e be the complete set $\{v \in V(G) : e \in E(P_v)\}$. For every claw Y in T , let K_Y be the complete set $\{v \in V(G) : P_v \text{ contains two spokes of } Y\}$.

Theorem 2 [4] *Let $\langle \mathcal{P}, T \rangle$ be an *EPT* representation of G . If C is a clique of G then either there is an edge $e \in E(T)$ such that $C = K_e$ or there is a claw Y in T such that $C = K_Y$.*

In the former case, when there exists e such that $C = K_e$, the clique C is called an **edge-clique**, otherwise C is called a **claw-clique**. See Figure 2. Notice that the condition of being an edge-clique or a claw-clique depends on the given representation. Clearly, in a Helly *EPT* representation every clique is an edge-clique. We say that three paths of a given *EPT* representation $\langle \mathcal{P}, T \rangle$ **form a claw** if there exists a claw Y of T such that every pair of

spokes of Y is contained by some of the paths. Clearly, there is claw-clique if and only if three paths form a claw.

If $S \subseteq V(G)$ then $G - S$ denotes the graph induced in G by $V(G) \setminus S$. When S contains a unique vertex v , we write simply $G - v$.

3 Gates and multiples

A clear corollary of Theorem 1 is that every chordless cycle C_k with $k > h \geq 3$ is an obstruction for the class $[h, 2, 2]$. In [6], Golumbic et al. wonder if besides cycles there are other *EPT* forbidden induced subgraphs for this class. In [1], answering negatively the previous question, we described for every $h > 4$ an *EPT* graph F_h which has no induced cycles of size k for every $k > h$, but it does not admit an *EPT* representation on a host tree with maximum degree less than or equal to h . The graphs introduced in the following definition generalize the graphs F_h . In Section 4, we obtain a total characterization of Helly $[h, 2, 2]$ graphs using them.

We say that two graphs G and G' are **disjoint** if $V(G) \cap V(G') = \emptyset$. The **union** of G and G' is the graph H with $V(H) = V(G) \cup V(G')$ and $E(H) = E(G) \cup E(G')$.

Definition 3 *The following graphs are called **gates**.*

- Every chordless cycle C_n with $n \geq 4$ is a gate;
- If G is a gate, C and C' are disjoint cliques of G , and $P : (v_1, \dots, v_l)$ with $l \geq 2$ is a chordless path disjoint from G , then the union of G and P plus all edges between v_1 and the vertices of C , and all edges between v_l and the vertices of C' is a gate;
- There are no more gates.

*If the number of cliques of a gate G is k then we say that G is a **k -gate**.*

In Figure 3 we offer some examples of gates.

Lemma 4 *If G is a k -gate then $G \in \text{Helly } [k, 2, 2]$. Furthermore, G admits a Helly $(k, 2, 2)$ -representation on a host tree that is a star.*

PROOF. We proceed by induction. Clearly the statement holds for C_k .

If G is not a cycle, then G is obtained from an m -gate H using disjoint cliques C and C' of H and a path $P : (v_1, v_2, \dots, v_l)$ with $l \geq 2$ disjoint from H . Notice that $m + (l - 1) = k$. Let $\langle \mathcal{P}, T \rangle$ be a Helly $(m, 2, 2)$ -representation of H with

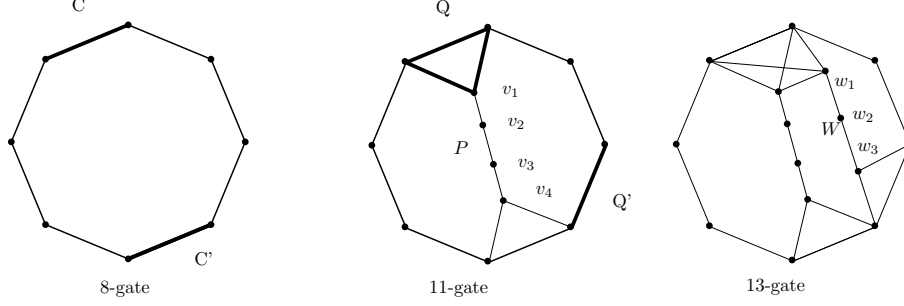


Fig. 3. Some examples of gates. From left to right, the second gate is obtained from the first using the bold cliques C and C' and the path $P : (v_1, v_2, v_3, v_4)$. The third gate is obtained from the second using the bold cliques Q and Q' and the path $W : (w_1, w_2, w_3)$.

T a star. We can assume that T has m spokes. Let e and e' be spokes of T such that $C = K_e$ and $C' = K_{e'}$. Denote by T' the star that is obtained by adding $l - 1$ spokes e_1, \dots, e_{l-1} to T . Let P_{v_1} be the subpath of T' defined by the edges e and e_1 . For $2 \leq i \leq l - 1$ let P_{v_i} be the subpath of T' defined by the edges e_{i-1} and e_i ; and let P_{v_l} the one defined by the edges e_{l-1} and e' .

Thus $\langle \mathcal{P}', T' \rangle$ is a Helly $(k, 2, 2)$ -representation of G , where \mathcal{P}' is the family \mathcal{P} plus the paths P_{v_i} for $1 \leq i \leq l$. \square

Lemma 5 *If G is a gate and $v \in V(G)$, then v belongs to exactly two cliques of G . In addition, if C_1 and C_2 are those cliques then $C_1 \cap C_2 = \{v\}$.*

PROOF. We proceed by induction. Clearly the statement holds for chordless cycles.

Let G be a gate obtained from another gate H , using disjoint cliques C and C' of H and a chordless path $P : (v_1, \dots, v_l)$ with $l \geq 2$ disjoint from H . Notice that the cliques of G are:

- the cliques of H other than C and C' ;
- the cliques of P , i.e. $\{v_i, v_{i+1}\}$ for $1 \leq i \leq l - 1$;
- $C \cup \{v_1\}$; and
- $C' \cup \{v_l\}$.

The proof follows easily from the fact that H satisfies the statement. \square

Lemma 6 *Let v be a vertex of a gate G , C_1 and C_2 cliques of G such that $C_1 \cap C_2 = \{v\}$, and $W : (w_1, \dots, w_t)$ a chordless path disjoint from G with $t \geq 2$. Then, the graph G' union of $G - v$ and W plus all edges between w_1 and the vertices of $C_1 - \{v\}$ and all edges between w_t and the vertices of $C_2 - \{v\}$ is a gate.*

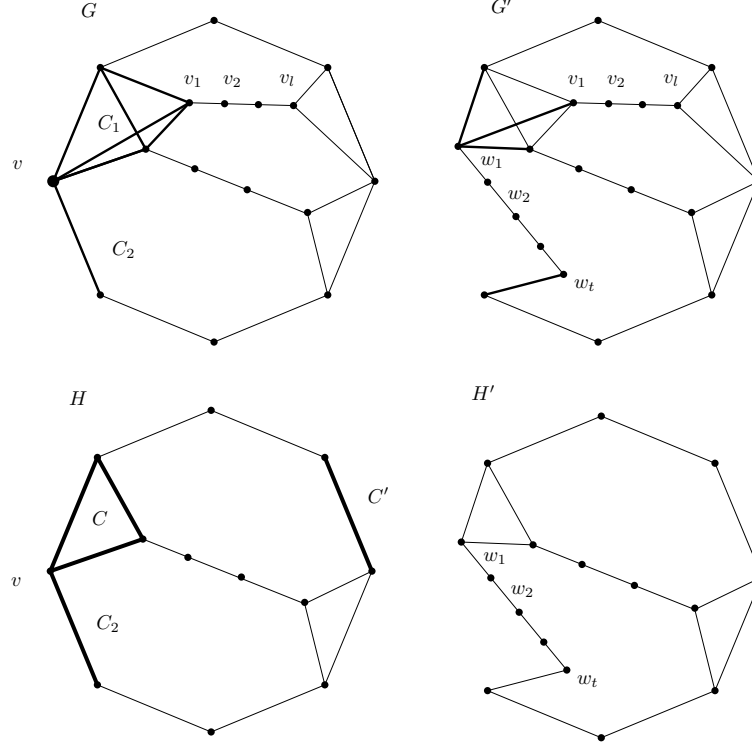


Fig. 4. An example following the proof of Lemma 6.

PROOF. We proceed by induction. Clearly the statement holds for chordless cycles.

Assume G is a gate obtained from another gate H , using disjoint cliques C and C' of H and a chordless path $P : (v_1, \dots, v_l)$ with $l \geq 2$ disjoint from H . If v is one of the vertices of P then the proof is direct and simple.

If v is a vertex of C (see Figure 4), we can assume that $C_1 = C \cup \{v_1\}$ and C_2 is a clique of G different from $C' \cup \{v_l\}$, which means that in H the vertex v is the intersection between the cliques C and C_2 . Thus, by the inductive hypothesis, the graph H' obtained from the union of $H - v$ and W plus all edges between w_1 and the vertices of $C - \{v\} = C_1 - \{v, v_1\}$ and all edges between w_t and the vertices of $C_2 - \{v\}$ is a gate. Since the path P is disjoint from H' , and $(C_1 - \{v_1, v\}) \cup \{w_1\}$ and C' are disjoint cliques of H' , thus, by the recursive definition of gate, the union of H' and P plus all edges between v_1 and the vertices of $(C_1 - \{v_1, v\}) \cup \{w_1\}$, and all edges between v_l and the vertices of C' is a gate. The proof follows from the fact that this is the same graphs G' depicted in the statement of the theorem.

If v is a vertex of C' or if $v \in V(H) - (C \cup C')$ the proof is analogous. \square

Golumbic and Jamison proved that (see Theorem 1) chordless cycles admit a *unique EPT* representation called pie. In what follow, generalizing that

result, we introduce the definition of multiple and prove that also gates admit a *unique EPT* representation.

Definition 7 A **multiple of size k** in an *EPT* representation $\langle \mathcal{P}, T \rangle$ is a star subgraph of T with central vertex q and neighbors q_1, \dots, q_k and a subfamily \mathcal{P}' of \mathcal{P} such that:

- (1) if $P \in \mathcal{P}'$ then $|V(P) \cap \{q_1, q_2, \dots, q_k\}| = 2$ (every path contains two spokes of the star);
- (2) if $i \neq j$ then $|\{P \in \mathcal{P}' : \{q_i, q_j\} \subseteq V(P)\}| \leq 1$ (no two paths contain the same two spokes);
- (3) if $1 \leq i \leq k$ then $|\{P \in \mathcal{P}' : \{q, q_i\} \subseteq V(P)\}| \geq 2$ (every spoke of the star is contained by at least two paths);
- (4) no three paths of \mathcal{P}' form a claw.

Observe that every pie is a multiple. The following theorem generalize Theorem 1.

Theorem 8 Let $\langle \mathcal{P}, T \rangle$ be an *EPT* representation of G . If G contains a k -gate then $\langle \mathcal{P}, T \rangle$ contains a multiple of size k whose paths are in one-to-one correspondence with the vertices of the gate.

PROOF. Let $\langle \mathcal{P}, T \rangle$ be an *EPT* representation of G whit $\mathcal{P} = (P_v)_{v \in V(G)}$. We can assume, without loss of generality, that G is a k -gate. We proceed by induction. If G is a chordless cycle C_k then, by Theorem 1, $\langle \mathcal{P}, T \rangle$ contains a pie of size k and the proof follows.

If G is not a cycle, then G is obtained from an m -gate H using disjoint cliques C and C' of H and a path $P : (v_1, v_2, \dots, v_l)$ with $l \geq 2$ disjoint from H . Notice that $m + (l - 1) = k$. By inductive hypothesis, $\langle \mathcal{P}, T \rangle$ contains a multiple of size m formed by a star subgraph S of T and the path subfamily $\mathcal{P}' = (P_v)_{v \in V(H)}$.

Let S be the star with center q and leaves q_1, \dots, q_m . By condition (4) in Definition 7, no three paths of \mathcal{P}' form a claw, then there exists a spoke of S , say $e_1 = qq_1$, such that $C \subseteq K_{e_1}$; and there exists another spoke, without loss of generality say $e_2 = qq_2$, such that $C' \subseteq K_{e_2}$. Even more, by condition (2), e_1 and e_2 are the only spokes of S satisfying the described property.

Let d be the minimum distance in H between a vertex of C and a vertex of C' . Clearly, $d \geq 1$. Chose vertices $u \in C$ and $u' \in C'$ such that the distance between them in H is d . Let $(u, u_1, u_2, \dots, u_{d-1}, u')$ be a shortest path in H between u and u' . Notice that $u, u_1, u_2, \dots, u_{d-1}, u', v_l, v_{l-1}, \dots, v_2, v_1$ induce a cycle in G of size $d + l + 1 \geq 4$. By Theorem 1, in $\langle \mathcal{P}, T \rangle$ there is a pie corresponding to this cycle. Let S' be the star subgraph of T used by this pie. Notice that the center of S' must be the same vertex q of T . Even more, since

the vertex v_1 of P is adjacent to all vertices in C , the vertex v_l is adjacent to all vertices in C' , and there are no other adjacencies between vertices of P and H , we have that S' has $l - 1$ spokes that are not spokes of S . The remaining $(d + l + 1) - (l - 1) = d + 2$ spokes of S' are also spokes of S . Therefore the union of S and S' is a star subgraph of T with center q and $m + l - 1 = k$ spokes. Now it is not difficult to check that \mathcal{P} forms a multipie around the star $S \cup S'$, and the proof follows. \square

4 Forbidden induced subgraphs for Helly EPT graphs on bounded degree trees

The goal of this section is Theorem 9 below. We prove that gates are the only subgraphs which force the use a host a tree with large enough degree in every Helly *EPT* representation of a graph.

Theorem 9 *Let G be a Helly EPT graph and $h \geq 3$. Then, $G \notin \text{Helly } [h, 2, 2]$ if and only if there exists $k > h$ such that G has a k -gate as induced subgraph.*

PROOF. We will prove the direct implication, the converse follows from Theorem 8 and the fact that $\text{Helly } [h, 2, 2] \subseteq [h, 2, 2]$.

Assume that G is a Helly *EPT* graph which does not admit a Helly $(h, 2, 2)$ -representation. Let d be the smallest positive integer such that $G \in \text{Helly } [d, 2, 2]$. Clearly, $d > h$. Let $\langle \mathcal{P}, T \rangle$ be a Helly $(d, 2, 2)$ -representation of G minimizing the number of vertices of the host tree T with degree d .

Claim 10 *We can assume that if $q \in V(T)$ is the end vertex of a path $P \in \mathcal{P}$ then $d_T(q) \leq 2$.*

PROOF. If it is not the case, by subdividing every edge of T (and consequently every edge of every path of \mathcal{P}) in three parts, and after that shortening every path of \mathcal{P} by removing its two end vertices, we obtain the desired representation. \square

Let $q_0 \in V(T)$ be a vertex with degree d and call q_1, \dots, q_d to its neighbors. Denote by H the subgraph of G induced by the vertices v such that $q_0 \in V(P_v)$.

Claim 11 *The subgraph H contains an induced cycle of length at least 4.*

PROOF. Let $P = (v_1, \dots, v_l)$ be the longest induced path in H and assume, without loss of generality, that $\{q_i, q_0, q_{i+1}\} \subseteq V(P_{v_i})$, for all $i : 1, \dots, l$. Notice that $2 \leq l \leq d - 1$.

Suppose, in order to derive a contradiction, that every path of \mathcal{P} containing q_0q_{l+1} also contains q_0q_l . Then, we can modify (as explained below) the representation $\langle \mathcal{P}, T \rangle$ to obtain a new Helly $(d, 2, 2)$ -representation of G on a host tree with less vertices of degree d , contrary to our assumption. Indeed, to obtain the new representation do:

subdivide the edge q_0q_l adding a new vertex \tilde{q}_l ;

remove the edge q_0q_{l+1} and do q_{l+1} adjacent to \tilde{q}_l ;

in the paths of \mathcal{P} containing the edge q_0q_{l+1} , replace the vertex q_0 and the edges q_0q_{l+1} and q_0q_l by the vertex \tilde{q}_l and the edges \tilde{q}_lq_{l+1} and \tilde{q}_lq_l , respectively;

no other path is modified except for the fact of subdividing the edge q_0q_l if necessary.

Therefore, there must exist $1 \leq j \leq d$, $j \neq l, l + 1$, and a vertex x of H such that $\{q_j, q_0, q_{l+1}\} \subseteq V(P_x)$. Clearly, $x \notin V(P)$.

If $j > l + 1$, then $V(P) \cup \{x\}$ induces a path of H longer than P , which contradicts the election of P .

If $j = l - 1$, then P_x , $P_{v_{l-1}}$ and P_{v_l} violate the Helly property, which contradicts the fact that this is a Helly *EPT* representation of G .

Thus $j \leq l - 2$. This implies that H contains the cycle induced by the vertices $\{v_j, \dots, v_{l-1}, v_l, x\}$, as we wanted to prove. \square

It follows from the previous Claim 11, that H has at least an induced gate. Let R be a biggest induced gate in H , say that R is a k -gate, and assume without loss of generality, that the multipie corresponding to the vertices of R use the star with edges $\{q_0q_1, \dots, q_0q_k\}$ (see Lemma 8).

We will prove that $k = d$. Since $d > h$, the proof follows.

Clearly, $k \leq d$. Suppose, in order to derive a contradiction, that $k < d$.

Since G is connected there must exists a vertex y such that the path P_y uses one of the edges q_0q_1, \dots, q_0q_k and an edge q_0q_i for some $k < i \leq d$. Without loss of generality, we can assume that $\{q_k, q_0, q_{k+1}\} \subseteq V(P_y)$.

If all paths containing the edge q_0q_{k+1} also contain the edge q_0q_k , then (as we did before) we can modify the representation $\langle \mathcal{P}, T \rangle$ to obtain a new representation of G on a host tree with fewer vertices of degree d , contrary to assumption.

So, there exists a vertex z and $j \neq k, k+1$ such that $\{q_j, q_0, q_{k+1}\} \subseteq V(P_z)$. Notice that y and z are adjacent and do not belong to the gate R .

Assume, in order to derive a contradiction, that $j \leq k-1$. Let C_k and C_j be the cliques of R corresponding to the edges q_0q_k and q_0q_j of T , respectively. Notice that C_k and C_j are disjoint, otherwise P_y , P_z and P_v violate the Helly property, where v is a vertex in the intersection. Using cliques C_k , C_j and the path $P : (y, z)$ disjoint from R , we obtain a $(k+1)$ -gate induced in H , which contradicts the election of R . Therefore, $j > k-1$. Since $j \neq k, k+1$, say $j = k+2$.

Denote by A the set of vertices $v \in V(H)$ such that P_v contains an edge q_0q_i for some $i \leq k$ and an edge $q_0q_{i'}$ for some $i' > k$. Notice that $y \in A$ and $z \notin A$. Let G_z be the connected component of $G - A$ containing the vertex z .

Clearly, if $v \in V(G_z) \cap V(H)$ then there exist i and i' , $k+1 \leq i < i' \leq d$ such that $\{q_i, q_0, q_{i'}\} \subseteq V(P_v)$, thus, without loss of generality, we can assume that there exists s , with $k+2 \leq s \leq d$, such that

$$V(G_z) \cap V(H) = \bigcup_{k+1 \leq i < i' \leq s} \{v \in V(G) : \{q_i, q_0, q_{i'}\} \subseteq V(P_v)\};$$

and for every $k+1 \leq i \leq s$

$$\text{there exists } v \in V(G_z) \cap V(H) \text{ such that } q_0q_i \in E(P_v). \quad (1)$$

Claim 12 *If $y' \in A$ and $P_{y'}$ contains an edge q_0q_t with $k+1 \leq t \leq s$ then $P_{y'}$ also contains the edge q_0q_k .*

PROOF. Assume, in order to derive a contradiction, that $q_0q_j \in E(P_{y'})$ with $1 \leq j < k$. Since $q_0q_t \in E(P_{y'})$ and $k+1 \leq t \leq s$, by (1), there exists $z' \in V(G_z) \cap V(H)$ adjacent to y' . We chose z' minimizing its distance to z in G_z (it could be $z' = z$). Let $P : (z, z_1, \dots, z')$ be a shortest zz' -path in G_z . It is clear that y' is adjacent to no vertex of P except z' . Notice also that $V(P) \cap V(R) = \emptyset$, and no vertex of P is adjacent to a vertex of R . So we will deal with the following cases:

- (a) y is adjacent to y' (in this case t must be equal to $k+1$).
- (b) y is non adjacent to y' but it is adjacent to some vertex of P besides z .
Thus, it must be z_1 and (y, z_1, \dots, z', y') is a chordless path.

- (c) y is neither adjacent to y' nor to a vertex of P besides z . Thus $(y, z, z_1, \dots, z', y')$ is a chordless path.

Let C_j and C_k be the cliques of R corresponding to the edges q_0q_j and q_0q_k , respectively. If C_j and C_k are disjoint, then, by Definition 3 a gate bigger than R can be obtained using these two cliques and the path described above depending on cases (a), (b) or (c). If C_j and C_k are non disjoint, then, by Lemma 6 a gate bigger than R can be obtained also using these two cliques and the path described above depending on cases (a), (b) or (c). It contradicts the fact that R is the biggest gate. \square

Finally, to end the proof of Theorem 9, we will describe below how to obtain a new Helly $(d, 2, 2)$ -representation $\langle \mathcal{P}', T' \rangle$ of G using a host tree T' with fewer vertices of degree d . This contradicts the fact that $\langle \mathcal{P}, T \rangle$ is a representation minimizing the number of vertices with degree d and the proof follows.

To obtain T' do:

subdivide the edge q_0q_k of T adding a new vertex \tilde{q}_k adjacent to q_0 and to q_k ;

for every $k+1 \leq i \leq s$, remove the edge q_0q_i and add the edge \tilde{q}_kq_i .

To obtain \mathcal{P}' do:

If $P \in \mathcal{P}$ and there exist $k+1 \leq i < j \leq s$ such that $\{q_i, q_0, q_j\} \in V(P)$, then replace P by the path P' obtained from P by removing the vertex q_0 and the edges q_0q_i and q_0q_j and adding the vertex \tilde{q}_k and the edges \tilde{q}_kq_i and \tilde{q}_kq_j .

If $P \in \mathcal{P}$ and there exists $k+1 \leq i \leq s$ such that $\{q_i, q_0\} \in V(P)$ and P is not in the previous case (then, by Claim 12 and the fact that G_z is a connected component of $G - A$, we have that $\{q_0, q_k\} \subseteq V(P)$), then replace P by the path P' obtained from P by removing the vertex q_0 and the edges q_0q_i and q_0q_k and adding the vertex \tilde{q}_k and the edges \tilde{q}_kq_i and \tilde{q}_kq_k .

No other path is modified except for the fact of subdividing the edge q_0q_k if necessary. \square

Corollary 13 *Helly $EPT \cap [h, 2, 2] = \text{Helly } [h, 2, 2]$ for any $h \geq 3$.*

PROOF. Clearly, $\text{Helly } [h, 2, 2] \subseteq \text{Helly } EPT \cap [h, 2, 2]$.

Assume, in order to derive a contradiction, that $G \in \text{Helly } EPT \cap [h, 2, 2]$ and $G \notin \text{Helly } [h, 2, 2]$. By 9, G contains a k -gate as induced subgraph for some $k > h$. Thus by Theorem 8, any EPT representation of G contains a multipie of size k . This contradicts the fact that $G \in [h, 2, 2]$. \square

5 Decomposition by clique separators and Complexity

A clique C of a connected graph G is a **separator** if $G - C$ (the subgraph induced by $V(G) \setminus C$) is not connected. An **atom** is a connected graph with no separators. In [10], a graph is progressively decompose by clique separators to obtain a **clique decomposition tree** with each leaf node being associated with an atom of G and each internal node being associated with a clique separator of G . The atoms of G are invariants. The clique decomposition can be computed in polynomial time. Both *EPT* graph and Helly *EPT* graphs are characterize by their clique decomposition tree. The characterization leads to an efficient algorithm to recognize Helly *EPT* graphs but does not to recognize *EPT* graphs.

Lemma 14 *If H is a Helly *EPT* atom with exactly $k \geq 4$ cliques then H has a k -gate as induced subgraph.*

PROOF. Assume, in order to derive a contradiction, that H has no k -gates, then it has no t -gates for any $t \geq k$. Thus, by Theorem 9, there exists $h \leq k-1$ such that $H \in \text{Helly } [h, 2, 2] - \text{Helly } [h-1, 2, 2]$. Let $\langle \mathcal{P}, T \rangle$ be a Helly $(h, 2, 2)$ -representation of H minimizing the number of edges of the host tree T , this implies that K_e is a clique of H for every $e \in E(T)$, moreover $|E(T)| = k$. On the other hand, since H is an atom, T must be a star (otherwise there exists an edge e of the host tree such that K_e is a cut clique). It follows that $h = k$, in contradiction with the fact that $h < k$. \square

Lemma 15 *Let H be an k -gate. If H is an induced subgraph of a graph G , then H is an induced subgraph of some atom of G .*

PROOF. It is enough to prove that a gate has no clique separators which follows trivially from the recursive definition of gates. \square

Theorem 16 *Let G be a Helly *EPT* graph and $h \geq 3$. Then, $G \in \text{Helly } [h, 2, 2]$ if and only if every atom of G has at most h cliques.*

PROOF. If $G \in \text{Helly } [h, 2, 2]$ then, by Theorem 9, G has no gates of size greater than h as induced subgraphs. Thus, by Lemma 14, G has no atoms with more than h cliques.

Conversely, assume, in order to obtain a contradiction, that $G \notin \text{Helly } [h, 2, 2]$. Thus, by Theorem 9, G has a k -gate H as induced subgraph, for some $k > h$. By Lemma 15, H is an induced subgraph of some atom of G . It implies that the atom has at least k cliques, which contradicts the assumption. \square

We will consider the following two problems, the first is posed for a given fixed $h \geq 4$.

RECOGNIZING HELLY $[h, 2, 2]$ GRAPHS

Input: A connected graph G .

Question: Does G belong to Helly $[h, 2, 2]$?

CHEAPEST REPRESENTATION

Input: A connected graph G .

Goal: Determine the minimum $h \geq 2$ such that $G \in \text{Helly } [h, 2, 2]$.

Clearly an efficient solution of the latter implies an efficient solution of the former.

Theorem 17 *The problem CHEAPEST REPRESENTATION is polynomial times solvable.*

PROOF. Using the efficient algorithm described in [10], determine whether the given graph G belongs to Helly EPT or not. If it does then determine for each atom G_i of G its number of cliques, say k_i . Notice that it can be done efficiently since the total number of cliques of a Helly EPT graphs G is at most $\lfloor \frac{3|V(G)|-4}{2} \rfloor$ [10]. Let k be the maximum k_i .

If $k \leq 3$, then every atom is chordal which implies $G \in \text{Chordal} \cap EPT = [3, 2, 2]$ (see [10] and [4]). Now test whether G is an interval graph or not and answer $h = 2$ in an affirmative case and $h = 3$ otherwise.

If $k \geq 4$, by Theorem 16, $G \in \text{Helly } [k, 2, 2]$ and $G \notin \text{Helly } [k-1, 2, 2]$, thus let $h = k$. \square

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